Second-Order Drag

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Let's take a look at a drag force $F = -bv^2$ acting on a body of mass m dropped from rest under the influence of gravitational acceleration g. From Newton's second law, we have:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv^2 \tag{1}$$

Separating variables and integrating, we have:

$$\int \frac{\mathrm{d}v}{mg - bv^2} = \int \frac{\mathrm{d}t}{m} \tag{2}$$

The right-hand side is simple enough; to solve the left-hand side, we make the substitutions:

$$\begin{aligned} \sqrt{mg}\cos\theta &= \sqrt{mg - bv^2} \\ mg\cos^2\theta &= mg - bv^2 \end{aligned} \tag{3}$$

And:

$$\begin{aligned} \sqrt{mg}\sin\theta &= \sqrt{b}v \\ \sqrt{\frac{mg}{b}}\cos\theta\,\mathrm{d}\theta &= \mathrm{d}v \end{aligned} \tag{4}$$

With a bit of rearranging and canceling, this gives us the equivalent integral:

$$\int \frac{\mathrm{d}\theta}{\cos\theta} = \sqrt{\frac{bg}{m}} \int \mathrm{d}t \tag{5}$$

We know that the anti-derivative of $\sec\theta$ is $\ln(\sec\theta + \tan\theta)$, so this becomes:

$$\begin{aligned} \ln(\sec\theta + \tan\theta) &= \sqrt{\frac{bg}{m}}t + C_1 \\ \sec\theta + \tan\theta &= Ce^{\sqrt{bg/m}\,t} \end{aligned} \tag{6}$$

From our original trigonometric substitutions, we can derive expressions for $\sec\theta$ and $\tan\theta$ in terms of v, giving:

$$\frac{\sqrt{mg} + \sqrt{b}v}{\sqrt{mg - bv^2}} = Ce^{\sqrt{bg/m}\,t} \tag{7}$$

Since the object is dropped, $v = 0$ at $t = 0$, so $C = 1$. Then, we square both sides to obtain:

$$\frac{\left(\sqrt{mg} + \sqrt{b}v\right)^2}{mg - bv^2} = e^{\sqrt{bg/m}\,2t} \tag{8}$$

$$\frac{\sqrt{mg} + \sqrt{b}v}{\sqrt{mg} - \sqrt{b}v} = e^{\sqrt{bg/m}\,2t} \tag{9}$$

Finally, we isolate v to obtain:

$$v = \sqrt{\frac{mg}{b}} \cdot \frac{e^{\sqrt{bg/m}\,2t} - 1}{e^{\sqrt{bg/m}\,2t} + 1}$$

$$v = \sqrt{\frac{mg}{b}} \cdot \frac{e^{\sqrt{bg/m}\,t} - e^{-\sqrt{bg/m}\,t}}{e^{\sqrt{bg/m}\,t} + e^{-\sqrt{bg/m}\,t}}$$

$$\boxed{v = \sqrt{\frac{mg}{b}} \tanh\left(\sqrt{\frac{bg}{m}}t\right)} \tag{10}$$

We immediately note a few things: as $t \to \infty$, $v \to \sqrt{mg/b}$. This terminal velocity agrees well with a simpler analysis with Newton's first law. We also see that as the object gets more massive, the terminal velocity also increases, as expected; as the proportionality constant for the drag force increases, terminal velocity decreases (drag force is getting "stronger"). For fun, let's graph v as a function of t for various values of b, with $g = 9.81\ \mathrm{m\,s^{-2}}$ and $m = 1$ kg: