Dropping a Magnet down a Coil of Wire

Eric Zheng

March 20, 2019

In this document, I would like to explore the lab we did in class today regarding the voltage induced by dropping a small magnet down a coil of wire. First, we will approximate the magnetic field of the magnet; then, we will use this field to determine the induced voltage as a function of time.

1 Magnetic Field

The magnetic field of a bar magnet is actually somewhat tricky. To make things easier, we will pretend that the bar magnet is actually a small, ideal solenoid and that all of the field lines coming out of this solenoid are directed along its axis. For the scope of this experiment, I think this will be a sufficient approximation.

In class, we used Ampere's law to show that the magnetic field strength inside a solenoid is $B = \mu_0 ni$, where μ_0 is the permeability of free space, n is the linear coil density, and i is the electric current. However, we happen to be interested in the magnetic field outside the solenoid at some position x along the axis.

To accomplish this, we can use the law of Biot-Savart to first find the magnetic field strength at some distance x along the axis of a single loop of current with radius a. Since we have actually done this as an exercise in class, I will not go into too much detail here except to state that the result is:

$$B_a(z) = \frac{\mu_0 i a^2}{2(a^2 + z^2)^{3/2}} \tag{1}$$

We can then treat the solenoid of length l as a collection of $N = nl$ loops and sum up the contributions of each loop. For an infinitesimal length dz, there will be $n\,dz$ loops, each contributing its own field B_a, which is given above. This suggests that, at a distance x from the end of the solenoid, the magnetic field strength is given by the integral:

$$B(x) = \int_x^{x+l} nB_a(z)\,dz \tag{2}$$

Which gives us a field:

$$\boxed{B(x) = \frac{\mu_0 ni}{2}\left(\frac{x+l}{\sqrt{a^2+(x+l)^2}} - \frac{x}{\sqrt{a^2+x^2}}\right)} \tag{3}$$

To convince ourselves that our derivation is, in fact, correct, we can use it to determine the magnetic field at some point inside the solenoid as well. Let us suppose that the point P is now at some distance x along the axis inside of the solenoid. All of the loops above P will now oppose the loops below it, suggesting the net field:

$$B(x) = \int_{x}^{l-x} nB_a(z)\, dz - \int_{l-x}^{l} nB_a(z)\, dz \tag{4}$$

The result is somewhat lengthy, so I won't list it here, but Fig. 1 shows the solution for a very long solenoid ($l = 1000$). We see that, as we get closer to the center (away from "fringe effects" on the ends), the magnetic field approaches the ideal case ($B = \mu_0 ni$); moreover, this occurs whatever the radius a happens to be.

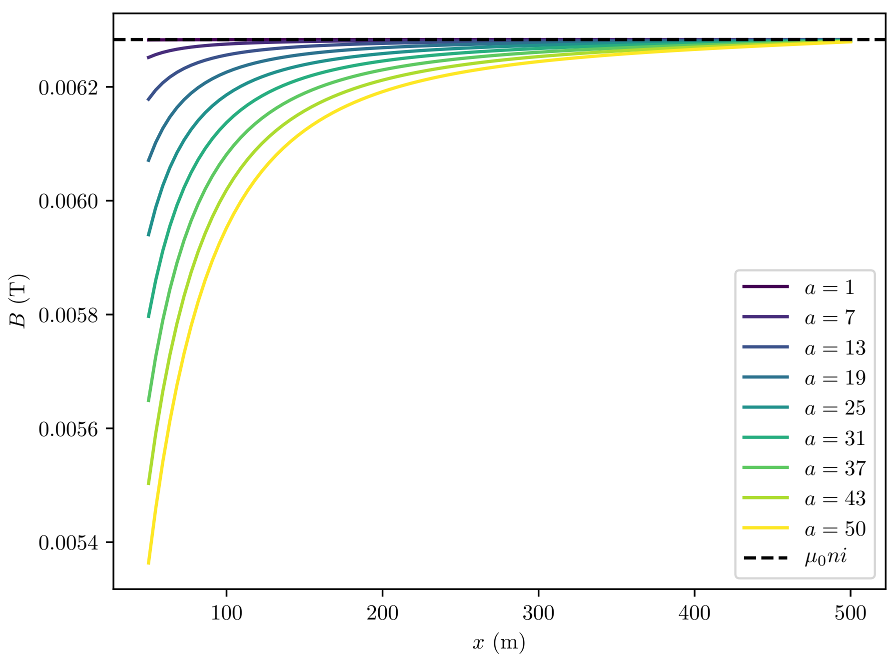

Figure 1: Solution to Eqn. 4 for different values of a. We have arbitrarily set $n = 500$ and $i = 10$.

2 Induced Voltage

Now that we have an expression for the magnetic field strength B at some distance x away from the end of the magnet, we are ready to find the induced voltage as the magnet is dropped through an outer coil. Let us suppose that this outer coil has length $L > l$ and $M = mL$ turns of wire. We have Faraday's law:

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{5}$$

Now all that remains is to determine the time derivative of the magnetic flux Φ. Assuming the field lines near the end of the magnet are all roughly normal to its surface, we can say that, through a single loop of the outer coil located a distance x from the end of the magnet, the flux is:

$$\Phi(x) = \iint \mathbf{B} \cdot d\mathbf{A} \approx \pi a^2 B(x) \tag{6}$$

By the chain rule, we have:

$$\frac{d\Phi}{dt} = \pi a^2 \frac{dB}{dx}\frac{dx}{dt} = \pi a^2 v \frac{dB}{dx} \tag{7}$$

Carrying out the differentiation, we have:

$$\mathcal{E}(x, v) = \frac{1}{2}\pi a^4 v \mu_0 n i \left(\frac{1}{(a^2 + x^2)^{3/2}} - \frac{1}{(a^2 + (x+l)^2)^{3/2}} \right) \tag{8}$$

Of course, since the magnet is falling, x and v are changing with time; that's where the time-dependence of $\mathcal{E}$ comes from. If we set the top of the outer coil as $x = 0$ and track the motion of the bottom of the magnet, we have: $x(t) \approx (1/2)gt^2$ and $v(t) \approx gt$, since the magnet starts at rest. We are pretending here that the magnet is in free fall. (This is not strictly correct because the induced current in the coil creates an opposing magnetic field that retards the motion of the magnet.)

Now note that the entire coil is made from a collection of loops, each of which produces a voltage $\mathcal{E}(z)$ as a function of its height z. We have to be careful here: any part of the coil that is above the magnet will produce a voltage opposing that in the parts of the coil below the magnet! (This follows from Lenz's law.) Since our outer coil density is m, an infinitesimal length of the solenoid will have $m\,dz$ coils, producing $\mathcal{E}(z) m\,dz$ voltage. This suggests that the net voltage induced as a function of time is given by the two integrals:

$$\mathcal{E}_{\text{net}} = \int_0^{x-l} m\mathcal{E}(z, v)\,dz - \int_0^{L-x} m\mathcal{E}(z, v)\,dz \tag{9}$$

Where x and v both vary with time. Note that this isn't entirely accurate when the magnet is near the ends of the coil (no voltage is produced in one side), but the error should be relatively small, so I won't bother correcting it with a piecewise definition of $\mathcal{E}_{\text{net}}$. Actually carrying out the integration and simplifying gives our desired result:

$$\boxed{\mathcal{E}_{\text{net}}(t) = Kv \left(\frac{L - x + l}{\sqrt{a^2 + (L - x + l)^2}} - \frac{L - x}{\sqrt{a^2 + (L - x)^2}} + \frac{x - l}{\sqrt{a^2 + (x - l)^2}} - \frac{x}{\sqrt{a^2 + x^2}} \right)} \tag{10}$$

Where we use the constant $K = \pi a^2 \mu_0 inm/2$ to avoid overflowing the text bounds even more. If we plot Eqn. 10, we get, sure enough, the two-peaked graph that we expect, shown in Fig. 2.

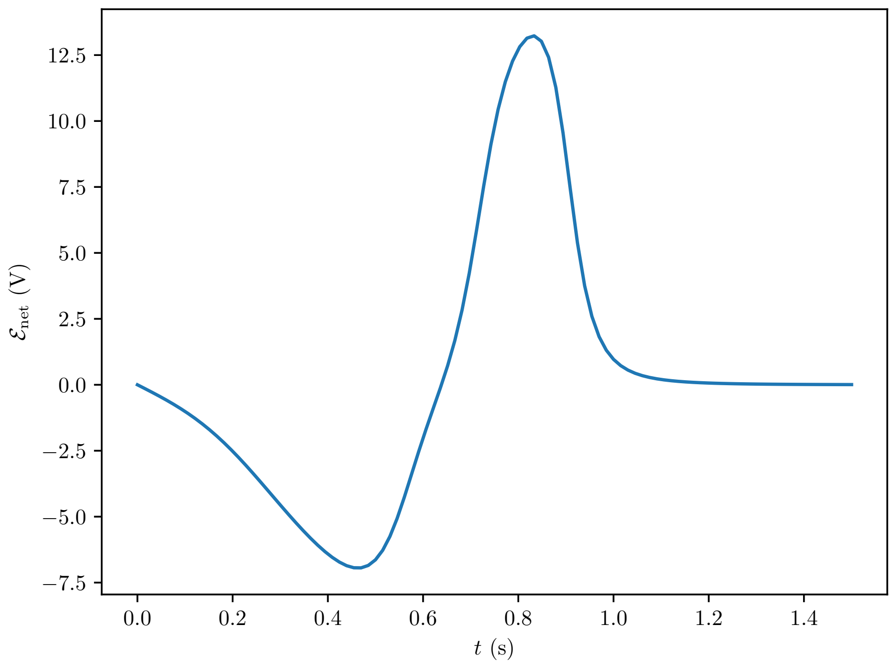

Figure 2: Plot of Eqn. 10, with $K = 1$, $l = 1.5$, $a = 0.5$, and $L = 2.5$ arbitrarily set. $g = 9.81$ is less arbitrarily set.

This graph has most of the characteristics that we'd expect. (And it's even dimensionally correct!) However, doing a quick numerical integration with `scipy`, I get:

$$\int_0^{1.5} \mathcal{E}_{\text{net}}(t)\, dt \approx 0.399 \tag{11}$$

Of course, the result should be 0. I suppose that, given the scale of things and the number of approximations we made along the way, this is a decently good answer, but it's possible that we made a mistake somewhere. Unfortunately, I have to go finish my homework now.