Projectile Range with Vertical Displacement

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Let us consider the range of a particle that undergoes a vertical displacement Δy and is launched with initial velocity v_0 at an angle θ. For convenience, we will set the origin $(0,0)$ at the place from which the projectile is launched, so Δx and Δy become x and y. We start with the constant-velocity kinematics equations:

$$\begin{aligned} x &= (v_0 \cos\theta)t \quad \Rightarrow \quad t = x/(v_0\cos\theta) \\ y &= (v_0 \sin\theta)t - (g/2)t^2 \end{aligned} \tag{1}$$

Substituting t, we see that:

$$\begin{aligned} y &= \frac{xv_0 \sin\theta}{v_0\cos\theta} - \frac{gx^2}{2v_0^2\cos^2\theta} \\ y &= x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta} \end{aligned} \tag{2}$$

If we like, we can rearrange and apply some trigonometric identities to solve for v_0 as:

$$v_0^2 = \frac{gx^2}{x\sin 2\theta - 2y\cos^2\theta} \tag{3}$$

From here, there are two ways to proceed: a messy numerical way and an elegant algebraic way. I'll present both, since the messy way is mine, but I wanted to share the elegant method I found on-line:

1 The Messy Way

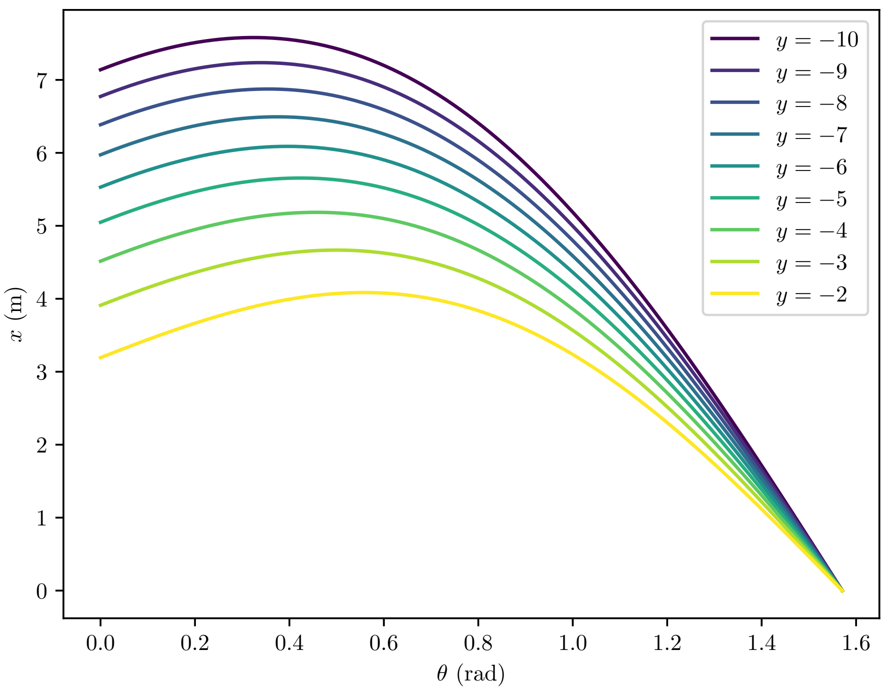
Really, we would ultimately like to have the range x as a function of the angle θ. Fortunately, this equation is quadratic in x:

$$gx^2 - (v_0^2\sin 2\theta)x + 2v_0^2 y\cos^2\theta = 0 \tag{4}$$

So we can apply the quadratic formula to obtain:

$$x = \frac{v_0^2\sin 2\theta + \sqrt{(v_0^2\sin 2\theta)^2 - 8gv_0^2 y\cos^2\theta}}{2g} \tag{5}$$

Let's take a look at the general shape of this graph, by setting $v = 5$ and $g = 9.81$: and plotting against various values of y:



Let's consider the graph for a moment. It seems to be about what we would expect: there is a maximum range at some angle between 0 and $\pi/4$, and when the angle is $\pi/2$ (the projectile is shot straight up), there is no horizontal displacement. We also see that as the vertical displacement increases, the angle that achieves the maximum horizontal displacement gets smaller.

Now let's consider what value of θ will maximize x. We can do this by differentiating with respect to θ and setting $\mathrm{d}x/\mathrm{d}\theta = 0$. Ths gives us the extremely nasty looking expression for our optimal angle θ:

$$\frac{v_0^2 \cos 2\theta}{g} + \frac{v_0^4 \sin 4\theta + 4gv_0^2 y \sin 2\theta}{2g\sqrt{(v_0^2 \sin 2\theta)^2 - 8gv^2 y \cos^2\theta}} = 0 \tag{6}$$

Honestly, you couldn't pay me enough money to go near that thing analytically; our best bet is to solve it numerically. We can do this, for example, with a short Python script:

```
import numpy as np
import scipy.optimize as opt

# these are the physical constants obtained during the lab
v = 3.079    # initial velocity, m/s
```

```
g = 9.81     # gravitational acceleration, m/s^2
y = -0.76    # vertical displacement, m

def dx_dt(t):
    # for sanity, we'll break the big, nasty equation into the form:
    # A + B / C
    A = (v ** 2) * np.cos(2*t) / g
    B = (v ** 4) * np.sin(4*t) + 4 * g * (v ** 2) * y * np.sin(2*t)
    C = 2*g*np.sqrt((v*v*np.sin(2*t))**2 - 8*g*v*v*y*(np.cos(t))**2)
    return A + B / C

print opt.newton_krylov(dx_dt, 0.2)
```

This gives us the angle $\boxed{\theta = 0.557473}$. This is about 31.9409°, which agrees extremely well with the $30° - 35°$ range obtained experimentally in class.

2 The Elegant Way

Unfortunately, the messy way doesn't give us much physical insight into the situation; it's pretty hard to visualize what's going on with all those trigonometric functions. But actually, there's a very elegant sleight of hand that can be performed in Equation 2 to give a much more beautiful solution. Recalling that $1/\cos^2\theta = \sec^2\theta = (1+\tan^2\theta)$, we can substitute $u = \tan\theta$ and $k = g/2v_0^2$ to obtain:

$$\begin{aligned} y &= ux - kx^2(1+u^2) \\ &= -kx^2 + ux - kx^2u^2 \end{aligned} \tag{7}$$

We can implicitly differentiate with respect to u to get:

$$0 = -2kx\frac{\mathrm{d}x}{\mathrm{d}u} + x + u\frac{\mathrm{d}x}{\mathrm{d}u} - 2kxu^2\frac{\mathrm{d}x}{\mathrm{d}u} - 2kx^2u \tag{8}$$

At the maximum, $\mathrm{d}x/\mathrm{d}u = 0$, so:

$$0 = x(1 - 2kxu) \tag{9}$$

This implies that $x = 1/2ku$. Substituting this back into Equation 7 and simplifying, we can then write:

$$u^2(1 - 4ky) = 1 \tag{10}$$

Recalling our original expressions for u and k, we finally see that to achieve maximum range, we should set:

$$\boxed{\theta = \tan^{-1}\frac{1}{\sqrt{1 - 2gy/v_0^2}}} \tag{11}$$

Using the values from our lab, we again get $\theta = 0.557473$ (n.b. we set y to be negative here, even though we set g to be positive). This now gives us better insight: as y gets larger in magnitude, the denominator grows, so the angle gets smaller and smaller.