Velocity of a Falling Rod

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Today in class, we found the angular velocity of a rod falling about one end, as shown in the figure below:

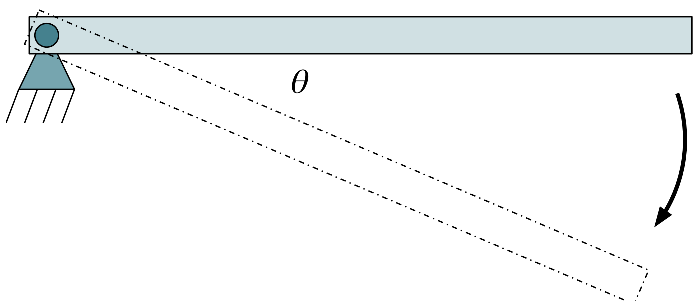

Figure 1: A rod fixed about its end and allowed to fall under the influence of gravity.

If the rod has mass m and length l, we want to find the angular velocity at the bottom (i.e. when $\theta = \pi/2$). The purpose of this document is to go over two methods: (1) the conservation of energy, as shown in class, and (2) Newton's laws of motion. Unsurprisingly, these two methods give identical results.

1 Conservation of Energy

When the rod is at the bottom, its center of mass has fallen an distance $l/2$. The corresponding change in potential energy is $U = mgl/2$. This lost potential energy is transformed into *rotational* kinetic energy of the form $K = I\omega^2/2$, where the moment of inertia I of a rod about its end is equal to $ml^2/3$. We can then say:

$$
\begin{aligned}
U &= K \\
\frac{1}{2}mgl &= \frac{1}{2}\frac{1}{3}ml^2\omega^2 \\
\Aboxed{\omega &= \sqrt{\frac{3g}{l}}}
\end{aligned} \tag{1}
$$

2 Newton's Laws of Motion

Alternatively, we can consider the equations of motion and use a little calculus to arrive at the same result. Newton's second law tells us that, for a rotating object, $\Sigma\tau = I\alpha$. The only force that exerts a torque on the rod is gravity, which always points directly down. With some geometry, we can determine that the component of gravity normal to the rod (and thus exerting a torque about the pivot) is $mg\cos\theta$, and it acts at a distance $l/2$ from the pivot. Then we have:

$$
\begin{aligned}
\Sigma\tau &= I\alpha \\
\frac{1}{2}mgl\cos\theta &= \frac{1}{3}ml^2\alpha \\
\alpha &= \frac{3g}{2l}\cos\theta
\end{aligned} \tag{2}
$$

Recall that α is the time derivative of ω. Then by the chain rule, we see that we can make the substitution:

$$
\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} \tag{3}
$$

The resulting equation can easily be separated and integrated:

$$
\begin{aligned}
\frac{3g}{2l}\cos\theta &= \omega\frac{d\omega}{d\theta} \\
\frac{3g}{2l}\cos\theta\,d\theta &= \omega\,d\omega \\
\int\frac{3g}{2l}\cos\theta\,d\theta &= \int\omega\,d\omega \\
\frac{3g}{2l}\sin\theta &= \frac{1}{2}\omega^2 + C
\end{aligned} \tag{4}
$$

But since $\omega = 0$ at $\theta = 0$, we have $C = 0$. Then solving for ω at $\theta = \pi/2$ is simple:

$$
\begin{aligned}
\frac{3g}{2l}\sin\theta &= \frac{1}{2}\omega^2 \\
\Aboxed{\omega &= \sqrt{\frac{3g}{l}}}
\end{aligned} \tag{5}
$$